NONLINEAR FLOWS AND RIGIDITY RESULTS ON COMPACT MANIFOLDS

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ABSTRACT. This paper is devoted to *rigidity* results for some elliptic PDEs and related interpolation inequalities of Sobolev type on smooth compact connected Riemannian manifolds without boundaries. Rigidity means that the PDE has no other solution than the constant one at least when a parameter is in a certain range. This parameter can be used as an estimate for the best constant in the corresponding interpolation inequality. Our approach relies in a nonlinear flow of porous medium / fast diffusion type which gives a clear-cut interpretation of technical choices of exponents done in earlier works. We also establish two integral criteria for rigidity that improve upon known, pointwise conditions, and hold for general manifolds without positivity conditions on the curvature. Using the flow, we are also able to discuss the optimality of the corresponding constant in the interpolation inequalities.

1. INTRODUCTION AND MAIN RESULTS

In the past decades there has been considerable activity in establishing sharp inequalities using maps or flows. The basic idea is to look for a flow on a function space along which a given functional converges to its optimal value, *i.e.*, one turns the idea of a Lyapunov function, known from dynamical systems theory, on its head. An example is furnished by the relatively recent proofs of the Brascamp-Lieb inequalities using non-linear heat flows in [17, 57, 26]. Using the same methods a new Brascamp-Lieb - type inequality on $\mathbb{S}^d$ was proved in [26]. Likewise the reverse Brascamp-Lieb inequalities can also be obtained in this fashion (see [14]). Another example is the proof of Lieb's sharp Hardy-Littlewood-Sobolev inequality given in [27] where a discrete map on a function space was constructed whose iterations drives the Hardy-Littlewood-Sobolev functional to its sharp value. Likewise, the sharp form of the Gagliardo-Nirenberg inequalities due to Dolbeault and Del Pino can be derived using the porous media flow (see [29] and [25] for the relation between the porous media equation and a special class of Hardy-Littlewood-Sobolev inequalities).

Closely related are the proofs of sharp inequalities using transportation theory. The earliest use of transportation theory to our knowledge was in Barthe's proof of the Brascamp-Lieb inequalities as well as their converse, in [13]. Transportation ideas were also applied in [32] for proving the sharp Gagliardo-Nirenberg inequalities and in [53] for proving sharp trace inequalities for Sobolev functions.

In this paper we invent a new type of porous media flow on Riemannian manifolds that allow us to give relatively straightforward proofs as well as generalizations of rigidity results of [21, 11, 49] for a class of non-linear equations. Before describing the flow, we discuss the rigidity results.

Throughout the paper we assume that $(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold of dimension $d \geq 1$, without boundary with Δ_g being the Laplace-Beltrami operator on $\mathfrak{M}$. For simplicity, we assume that the volume of $\mathfrak{M}$, $\mathrm{vol}(\mathfrak{M})$ is 1 and we denote by dv_g the volume element. We shall also denote by $\mathfrak{R}$ the Ricci tensor. Let λ_1 be the lowest positive eigenvalue of $-\Delta_g$. We shall use the notation $2^* := \frac{2\,d}{d-2}$ if $d \geq 3$, and $2^* := \infty$ if $d = 1$ or 2.

Let us start with results dealing with manifold with curvature bounded from below and define

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi, \xi)$$

Theorem 1. *Let $d \geq 2$ be an integer and assume that ρ is positive. If λ is a positive parameter such that*

$$\lambda \leq (1 - \theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \quad where \quad \theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2) + p - 1},$$

then for any $p \in (2, 2^*)$, the equation

$$(1) \qquad -\Delta_g v + \frac{\lambda}{p-2}\left(v - v^{p-1}\right) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$, which is constant and equal to 1.

Such a *rigidity* result has been established in [11, Inequality (1.11)] by D. Bakry and M. Ledoux (also see [39]), and in [49, Theorem 2.1] by J.R. Licois and L. Véron. An earlier version of it can be found in a work of M.-F. Bidaut-Véron and L. Véron, [21], with an estimate only in terms of ρ, which is not as good for general manifolds but coincides with the one given above for spheres. Earlier results can be found in [40]. Each of these contributions relies either on the Bochner-Lichnerovicz-Weitzenböck formula or on the *carré du champ* method, which are equivalent in the present case.

The case of the critical exponent $p = 2^*$ when $d \geq 3$ is not covered by Theorem 1 and requires more care, due to compactness issues and conformal invariance. For simplicity we shall restrict ourselves to the subcritical case in this paper, at least as far as rigidity results are concerned.

We shall now state a slightly more general result, which does not require that ρ is positive but only involves the constant

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1-\theta)\left(\Delta_g u\right)^2 + \frac{\theta\, d}{d-1}\, \mathfrak{R}(\nabla u, \nabla u) \right] dv_g}{\int_{\mathfrak{M}} |\nabla u|^2\, dv_g} .$$

Here θ is defined as in Theorem 1. If ρ is positive, it is not difficult to check (see Lemma 7) that

$$\lambda_\star \geq (1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} .$$

Here we shall simply assume that $\lambda_\star$ is positive but make no assumption on the sign (in the sense of quadratic forms) of the Ricci tensor $\mathfrak{R}$. Our first result generalizes Theorem 1 as follows.

Theorem 2. *With the above notations, if λ is such that*

$$0 < \lambda < \lambda_\star ,$$

then for any $p \in (1, 2) \cup (2, 2^*)$, Equation (1) has a unique positive solution in $C^2(\mathfrak{M})$, which is constant and equal to 1.

In this statement we include the case $p \in (1, 2)$. We may observe that $\lim_{p \to 1_+} \theta(p) = 0$, so that $\lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$, as soon as ρ is bounded (but eventually negative). Notice that if $\mathfrak{M} = \mathbb{S}^d$, then $\lambda_\star = \lambda_1 = d\,\rho/(d-1) = d$, $\rho = d-1$. In this case, the result of Theorem 1 is then optimal, but this was already known from [21, Theorem 6.1]. As we shall see in Proposition 13, we always have

$$\lambda_\star \leq \lambda_1$$

and (1) has non constant solutions for any $\lambda > \lambda_1$. We can actually give another integral criterion for rigidity that slightly improves the result of Theorem 2. Recalling that θ defined in Theorem 1 is given by

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} ,$$

let us define

$$\mathrm{Q}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u - \frac{(d-1)\,(p-1)}{\theta\,(d+3-p)}\left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d}\frac{|\nabla u|^2}{u}\right]$$

where $\mathrm{H}_g u$ denotes Hessian of u, and define

$$(2) \qquad \Lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{(1-\theta)\int_{\mathfrak{M}} \left(\Delta_g u\right)^2 dv_g + \frac{\theta\, d}{d-1}\int_{\mathfrak{M}} \left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] dv_g}{\int_{\mathfrak{M}} |\nabla u|^2\, dv_g} .$$

It is obvious that

$$\lambda_\star \leq \Lambda_\star ,$$

and it is not hard to see that there is equality in case of the sphere.

Theorem 3. *For any $p \in (1,2) \cup (2,2^*)$, Equation (1) has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0, \Lambda_\star)$, which is constant and equal to 1.*

Our approach relies on a nonlinear diffusion equation of porous media (fast diffusion) type. Its interest is that it simplifies the proofs of previously known results. It applies to equations which are not of power law type (see Theorem 14) and also provides an expression of the error term (see Corollary 15) in the interpolation inequality

$$(3) \qquad \|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2 \geq \frac{\lambda}{p-2} \left[\|v\|_{\mathrm{L}^p(\mathfrak{M})}^2 - \|v\|_{\mathrm{L}^2(\mathfrak{M})}^2 \right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M}) \,.$$

Notice that this inequality makes sense not only for $p \in (2,2^*)$ but also for $p \in (1,2)$ because in this last case we have $\|v\|_{\mathrm{L}^p(\mathfrak{M})} \leq \|v\|_{\mathrm{L}^2(\mathfrak{M})}$ by Hölder's inequality and $p - 2 < 0$. Inequality (3) has been stated in [21, Corollary 6.2] with $\lambda = d\,\rho/(d-1)$, but has also been established in the case of the sphere in [16] with $\lambda = d$ using spectral estimates and Lieb's sharp Hardy-Littlewood-Sobolev inequality [50]. We shall see in Proposition 13 that $\Lambda_\star \leq \lambda_1$. As a consequence of Theorem 3, we get the following result.

Theorem 4. *For any $p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, $p \in (1,2) \cup (2,\infty)$ if $d = 1$ or 2, Inequality (3) holds with $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$. Moreover, if $\Lambda_\star < \lambda_1$, then the best constant Λ is such that*

$$\Lambda_\star < \Lambda \leq \lambda_1 \,.$$

If $d = 1$, then $\Lambda = \lambda_1$.

While the case $p = 2^*$ is not covered in Theorem 3, inequality (3) holds in that case by taking the limit $p \to 2^*_-$ when $d \geq 3$. In the limit case $p = 1$, inequality (3) is nothing else but the Poincaré inequality which is consistent with the fact that $\lim_{p \to 1_+} \Lambda_\star(p) = \lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$. Using $u = 1 + \varepsilon\,\varphi$ as a test function where φ is an eigenfunction associated with the lowest positive eigenvalue of $-\Delta_g$, it is a classical result that the optimal constant in (3) is $\lambda \leq \lambda_1$. For the same reason, a minimum of

$$v \mapsto \|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2 - \frac{\lambda}{p-2} \left[\|v\|_{\mathrm{L}^p(\mathfrak{M})}^2 - \|v\|_{\mathrm{L}^2(\mathfrak{M})}^2 \right] \,,$$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$. Incidentally, such a minimization provides a non constant nonnegative solution of (1) (see [23, 37]). It is natural to wonder what happens when $p = 2$. The reader is invited to check that by taking the limit as $p \to 2$, one gets a logarithmic Sobolev inequality, which is of independent interest (see for instance [24] or [37] for more results in this direction and further references).

The main contribution of this paper is to introduce the flow

$$(4) \qquad u_t = u^{2-2\beta} \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) \,,$$

where we set

$$(5) \qquad \kappa = 1 + \beta\,(p-2) \,.$$

The function u which evolves according to (4) and the function v which appears in Theorems 2, 3 and Theorem 4 are related by $v = u^\beta$. The flow (4) contracts both sides of Inequality (3) and also

$$(6) \qquad \mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 \, dv_g + \frac{\lambda}{p-2} \left[\int_{\mathfrak{M}} u^{2\beta} \, dv_g - \left(\int_{\mathfrak{M}} u^{\beta p} \, dv_g \right)^{2/p} \right] \,.$$

This contraction property actually determines β in a way which is somewhat more transparent than in [21, 49] (see Section 3) and allows for simple generalizations, like the one stated in Theorem 14 concerning more general nonlinearities than just power laws. The choice of κ is determined by the condition that $\|v\|_{\mathrm{L}^p(\mathfrak{M})}$ is invariant under the action of the flow (see Lemma 8). The equation (4) is of the type of a porous media equation. Let us note in this context that Hamilton's Yamabe flow can be seen as the fast diffusion case of a porous media equation [59] but it is not clear whether there is any connection between this and (4).

This paper is organized as follows. In Section 2, we introduce two simple computations, which differ from the ones that can be found in [21, 11, 49]: results are stated in Lemmata 5 and 6. Section 3 is devoted to the evolution of the functional $\mathcal{F}$ given by (6) under the action of the flow defined by (4). Proofs of

Theorem 2 and Theorem 4 are given in Section 4. However, a simple inspection of the proofs allows to state an improved version of Theorem 4, with an integral remainder term. The corresponding results and some consequences have been collected in Section 5, as well as the proof of Theorem 3. The question of the optimality will also be discussed in Section 5. The use of the flow (4) shows that estimates are in general not optimal, although our approach relies only on global estimates (rather than on pointwise estimates). We stress the fact that global estimates allow us to consider the case of sign changing curvatures.

Before dealing with our method, let us give a brief list of papers which are relevant for our approach. We will only mention the ones which are closely related to our purpose and select in the huge literature corresponding to the various topics listed below the earliest contributions we are aware of, or some recent papers to which we refer for more details on the historical developments. The motivation for this work comes from the method developed by B. Gidas and J. Spruck in [40, Theorems B.1 and B.2] for proving *rigidity* and later improved first by M.-F. Bidaut-Véron and L. Véron in [21, Theorem 6.1], and then by D. Bakry and M. Ledoux in [11, Inequality (1.11)] and by J.R. Licois and L. Véron in [49, Theorem 2.1]. The choice of the exponents was somewhat mysterious in these papers while the choice of β and κ is rather straightforward and natural in our setting.

When $\mathfrak{M} = \mathbb{S}^d$, the *rigidity* result with optimal range in terms of the parameter λ was established in [21] and later, by other methods, in [16]. The case of the critical exponent was known much earlier since the inequality corresponds to the Sobolev inequality on the Euclidean space, after a stereographic projection: we can refer to [4, Corollaire 4], to [54] for issues dealing with conformal invariance, to [36] for results on the sphere, and to [37] for consequences in the spectral theory of Schrödinger operators.

For general manifolds, the so-called A-B problem, which amounts to determine the two best constants in interpolation inequalities, has attracted the attention of many authors and it is out of the scope of this paper to list all of them. Let us simply quote [5, 6, 47, 44, 43, 12] as a selection of contributions on the "second best constant" problem. The interested reader can also refer to [42] for a large overview of the topic. With some exceptions (see for instance [24, 35]), almost all related papers are concerned either with the case of critical exponents and related concentration issues or with the limiting case $p = 2$, where Inequality (3) has to be replaced by the logarithmic Sobolev inequality (see [38, 41, 56, 52] for early contributions to this topic).

Inequality (3) has been studied for $p \in (1, 2)$ and various probability measures, mostly defined on the Euclidean space: we may refer for instance to [9, 10, 45, 30, 35, 22]. The paper [35] by P. Deng and F. Wang is of particular interest, as it gives estimates of the constant λ in (3) for arbitrary $p \in (1, 2)$ using several methods, including the case of compact manifolds without assuming the positivity of the curvature. In the particular case of the sphere, the whole range $p \in (1, 2^*)$ was covered in the case of the ultraspherical operator in [18, 19]. Also see [36] for the use of the ultraspherical operator in the context of *rigidity* results.

Using flows on manifolds is a classical idea, including flows associated either to the heat equation or to porous media equations, or in relation with Ricci (see for instance [31]) and Yamabe flows (see [58, Appendix III] for an introduction). Another line of thought based on flows refers to the *carré du champ* method: see [9, 7, 1, 8] and [46] for a quite detailed review by M. Ledoux. In the framework of diffusion equations from the PDE point of view we primarily refer to [3] in the linear case, and to [29, 33, 28] for fast diffusion equations in the Euclidean space. This approach was later interpreted in terms of gradient flows with respect to Wasserstein's distance in [55], and then extended or reinterpreted in various contributions in the context of *entropy methods*. Such a direction of research is out of the scope of the present paper, although it has been a strong source of inspiration. Much less is known in the case of Riemannian manifolds: see [34, 51]. As far as we know, the flow defined by (4) has not been considered before, except in [36] when $\beta = 1$, or when $\kappa = 0$ and either $p = 2$ of $p = 2^*$.

2. SOME PRELIMINARY COMPUTATIONS

If $A_{i,j}$ and $B_{i,j}$ are two tensors we use the notation

$$A \cdot B := g^{i,m} g^{j,n} A_{i,j} B_{m,n} \quad \text{and} \quad \|A\|^2 := A \cdot A .$$

Here $g^{i,j}$ is the inverse of the metric tensor, *i.e.*, $g^{i,j} g_{j,k} = \delta_k^i$ where we used the Einstein summation convention and δ_k^i denotes the Kronecker symbol. Denote by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d} \Delta_g u$$

the trace free Hessian. The following lemma is well known, but since it is of some importance for what follows we give a short proof.

Lemma 5. *If $d \geq 2$ and $u \in C^2(\mathfrak{M})$, then we have*

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g = \frac{d}{d-1} \int_{\mathfrak{M}} \| \mathrm{L}_g u \|^2 \, dv_g + \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) \, dv_g \ .$$

Proof. Integrating the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

yields

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g = \int_{\mathfrak{M}} \|\mathrm{H}_g u\|^2 \, dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) \, dv_g \ ,$$

where, incidentally, we may remark that the right-hand side is nothing else than the integral of $\Gamma_2(u, u)$ in the *carré du champ* method: see for instance [11]. Next we notice that

$$[\mathrm{L}_g u] \cdot g = 0 \ ,$$

because $\Delta_g u$ is the trace of $\mathrm{H}_g u$, which implies

$$\int_{\mathfrak{M}} \| \mathrm{L}_g u \|^2 \, dv_g = \int_{\mathfrak{M}} \|\mathrm{H}_g u\|^2 \, dv_g - \frac{1}{d} \int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g \ ,$$

from which Lemma 5 follows. $\qquad\square$

The next lemma will be useful to express some important quantities as a square. Here we deal with the cross term.

Lemma 6. *If $u \in C^2(\mathfrak{M})$ is a positive function, then we have*

$$\int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, dv_g = \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g - \frac{2\,d}{d+2} \int_{\mathfrak{M}} [\mathrm{L}_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} \right] dv_g \ .$$

Proof. An integration by parts shows that

$$\int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, dv_g = \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g - 2 \int_{\mathfrak{M}} [\mathrm{H}_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} \right] dv_g \ ,$$

which equals

$$\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g - 2 \int_{\mathfrak{M}} [\mathrm{L}_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} \right] dv_g - \frac{2}{d} \int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, dv_g \ ,$$

by definition of $\mathrm{L}_g u$. Lemma 6 follows. $\qquad\square$

Recall that $(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold without boundary and consider the Poincaré inequality

$$(7) \qquad \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \geq \lambda_1 \int_{\mathfrak{M}} |u - \overline{u}|^2 \, dv_g \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M}) \quad \text{such that} \quad \overline{u} = \int_{\mathfrak{M}} u \, dv_g \ ,$$

with optimal constant $\lambda_1 > 0$. By standard compactness results, it is known that λ_1 can be characterized as the minimum of $\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g / \int_{\mathfrak{M}} |u - \overline{u}|^2 \, dv_g$ on all nonconstant functions $u \in \mathrm{H}^1(\mathfrak{M})$ and is achieved by a function $\varphi \in \mathrm{H}^1(\mathfrak{M})$ such that $\|\varphi\|_{\mathrm{L}^2(\mathfrak{M})} = 1$ and $\overline{\varphi} = 0$.

The third lemma of this section is a standard result, that is nonetheless crucial in our approach. For completeness, let us give a statement and a short proof.

Lemma 7. *With the above notations, we have*

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \quad \forall \, u \in \mathrm{H}^2(\mathfrak{M}) \, .$$

Moreover, λ_1 is the optimal constant in the above inequality.

Proof. A simple computation based on the Cauchy-Schwarz inequality

$$\left(\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \right)^2 = \left(\int_{\mathfrak{M}} \Delta_g u \, (u - \overline{u}) \, dv_g \right)^2 \leq \int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g \int_{\mathfrak{M}} |u - \overline{u}|^2 \, dv_g$$

shows that

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g \geq \frac{\left(\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \right)^2}{\int_{\mathfrak{M}} |u - \overline{u}|^2 \, dv_g}$$

and one observes using (7) that

$$\frac{\left(\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \right)^2}{\int_{\mathfrak{M}} |u - \overline{u}|^2 \, dv_g} \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \, ,$$

which concludes the proof. Notice that optimality in the Cauchy-Schwarz inequality means that $\Delta_g u$ and $(u - \overline{u})$ are collinear, that is, $(u - \overline{u})$ is an eigenfunction. It is then clear that equality holds if $u = \varphi$, an eigenfunction associated with the first positive eigenvalue λ_1. □

As a consequence of Lemmata 5 and 7, we find that

$$\lambda_1 = \inf \frac{\int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g}{\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g} \geq \frac{d}{d-1} \inf \frac{\int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) \, dv_g}{\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g} \, ,$$

thus establishing the Lichnerowicz estimate: $\lambda_1 \geq \frac{d \, \rho}{d-1}$ (see [48, p. 135] for the original statement, and [20, p. 179] for the Obata-Lichnerowicz theorem).

3. A POROUS MEDIA FLOW

This section is devoted to the study of the functional defined in (6) and its evolution under the action of the flow defined by (4), which is an equation of *porous media* type (the exponent β is in some cases smaller than 1 and one should then speak of *fast diffusion*: see discussion at the end of this section). We will expose the computations without taking care of regularity issues that are requested to justify integrations by parts and give afterwards a sketch of the proofs and regularizations that are eventually needed. The first statement explains how the coefficient κ is chosen.

Lemma 8. *With κ given by (5), the functional $u \mapsto \int_{\mathfrak{M}} u^{\beta \, p} \, dv_g$ remains constant if u is a smooth positive solution of (4).*

Proof. Computing the time derivative, we get that

$$\frac{d}{dt} \int_{\mathfrak{M}} u^{\beta \, p} \, dv_g = \beta \, p \int_{\mathfrak{M}} u^{1 + \beta \, (p-2)} \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) dv_g = 0$$

by the choice of κ. □

Next we compute the time derivative of the remaining terms in the functional (6).

Lemma 9. *If if u is a smooth positive solution of (4), the following identity holds*

$$\frac{1}{2 \, \beta^2} \frac{d}{dt} \mathcal{F}[u] = - \int_{\mathfrak{M}} \left[(\Delta_g u)^2 + (\kappa + \beta - 1) \, \Delta_g u \, \frac{|\nabla u|^2}{u} + \kappa \, (\beta - 1) \, \frac{|\nabla u|^4}{u^2} \right] dv_g + \lambda \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \, .$$

Proof. By Lemma 8, we only have to compute $\frac{d}{dt} \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 \, dv_g$ and $\frac{d}{dt} \int_{\mathfrak{M}} u^{2\beta} \, dv_g$. We get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 \, dv_g &= -\beta \int_{\mathfrak{M}} \Delta_g(u^\beta) \, u^{1-\beta} \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) dv_g \\
&= -\beta^2 \int_{\mathfrak{M}} \left(\Delta_g u + (\beta-1) \frac{|\nabla u|^2}{u} \right) \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) dv_g \\
&= -\beta^2 \int_{\mathfrak{M}} \left[(\Delta_g u)^2 + (\kappa + \beta - 1) \Delta_g u \frac{|\nabla u|^2}{u} + \kappa (\beta-1) \frac{|\nabla u|^4}{u^2} \right] dv_g
\end{aligned}
$$

and

$$
\begin{aligned}
(8) \qquad \frac{1}{2} \frac{d}{dt} \int_{\mathfrak{M}} u^{2\beta} \, dv_g &= \beta \int_{\mathfrak{M}} u^{2\beta-1} \, u^{2-2\beta} \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) dv_g \\
&= \beta (\kappa - 1) \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \\
&= \beta^2 (p-2) \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \; ,
\end{aligned}
$$

which completes the proof. $\qquad \square$

Notice that for any $\theta \in (0,1)$ we can write the result of Lemma 9 as

$$
\frac{1}{2 \beta^2} \frac{d}{dt} \mathcal{F}[u] = -(1-\theta) \int_{\mathfrak{M}} (\Delta_g u)^2 \, dv_g - \mathcal{G}[u] + \lambda \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \; ,
$$

where

$$
\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta (\Delta_g u)^2 + (\kappa + \beta - 1) \Delta_g u \frac{|\nabla u|^2}{u} + \kappa (\beta - 1) \frac{|\nabla u|^4}{u^2} \right] dv_g \; .
$$

We shall now introduce a key quantity for our method. For any positive function $u \in C^2(\mathfrak{M})$ set

$$
\mathrm{Q}_g^\theta u := \mathrm{L}_g u - \frac{1}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right] \; .
$$

Notice that Q_g^θ is a matrix valued, traceless quantity. The exponent β in the flow (4) and the value of $\theta \in [0,1]$ will be chosen below in this section.

Lemma 10. *Assume that $d \geq 2$. With the above notations, any positive function $u \in C^2(\mathfrak{M})$ satisfies the identity*

$$
\mathcal{G}[u] = \frac{\theta \, d}{d-1} \left[\int_{\mathfrak{M}} \|\mathrm{Q}_g^\theta u\|^2 \, dv_g + \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, dv_g \right] - \mu \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g
$$

with $\mu := \frac{1}{\theta} \left(\frac{d-1}{d+2} \right)^2 (\kappa + \beta - 1)^2 - \kappa(\beta-1) - (\kappa + \beta - 1) \frac{d}{d+2}$.

Proof. By Lemma 5 and 6, we know that

$$
\begin{aligned}
\mathcal{G}[u] &= \frac{\theta \, d}{d-1} \left[\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2 \, dv_g + \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, dv_g \right] \\
&\quad + (\kappa + \beta - 1) \left[\frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g - \frac{2d}{d+2} \int_{\mathfrak{M}} [\mathrm{L}_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} \right] dv_g \right] + \kappa(\beta-1) \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g
\end{aligned}
$$

which can be arranged into

$$
\begin{aligned}
\mathcal{G}[u] &= \frac{\theta \, d}{d-1} \left[\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2 \, dv_g - \frac{2}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \int_{\mathfrak{M}} [\mathrm{L}_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} \right] dv_g \right] \\
&\quad + \left(\kappa(\beta-1) + (\kappa + \beta - 1) \frac{d}{d+2} \right) \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, dv_g + \frac{\theta \, d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, dv_g \; .
\end{aligned}
$$

Noting again that $[\mathrm{L}_g u] \cdot g = 0$, we can rewrite the first line as

$$
\frac{\theta \, d}{d-1} \left[\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2 \, dv_g - \frac{2}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \int_{\mathfrak{M}} [\mathrm{L}_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right] dv_g \right] \; ,
$$

which, by completing the square, turns into

$$\frac{\theta\,d}{d-1}\int_{\mathfrak{M}}\|Q_g^\theta u\|^2\,dv_g - \frac{1}{\theta}\left(\frac{d-1}{d+2}\right)^2(\kappa+\beta-1)^2\int_{\mathfrak{M}}\frac{|\nabla u|^4}{u^2}\,dv_g$$

because

$$\int_{\mathfrak{M}}\left\|\frac{\nabla u\otimes\nabla u}{u}-\frac{g}{d}\frac{|\nabla u|^2}{u}\right\|^2\,dv_g = \left(1-\frac{1}{d}\right)\int_{\mathfrak{M}}\frac{|\nabla u|^4}{u^2}\,dv_g\;.$$

This completes the proof of Lemma 10. $\square$

Now let us explain how β and θ are chosen. Up to now, we have shown that

$$\frac{1}{2\,\beta^2}\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathfrak{M}}(\Delta_g u)^2\,dv_g - \frac{\theta\,d}{d-1}\left[\int_{\mathfrak{M}}\|Q_g^\theta u\|^2\,dv_g + \int_{\mathfrak{M}}\mathfrak{R}(\nabla u,\nabla u)\,dv_g\right]$$
$$+\,\mu\int_{\mathfrak{M}}\frac{|\nabla u|^4}{u^2}\,dv_g + \lambda\int_{\mathfrak{M}}|\nabla u|^2\,dv_g$$

where, after taking into account (5), μ is given by

$$\mu = \frac{1}{\theta}\left(\frac{d-1}{d+2}\right)^2\beta^2\,(p-1)^2 - \left(1+\beta\,(p-2)\right)(\beta-1) - \beta\,(p-1)\frac{d}{d+2}$$
$$= \left(\frac{1}{\theta}\left(\frac{d-1}{d+2}\right)^2(p-1)^2 - (p-2)\right)\beta^2 - 2\frac{d+3-p}{d+2}\beta+1\;.$$

The goal is to find values of β for which μ is equal to zero, thus maximizing the coefficients of the other terms. Unless $\theta\,(p-2) = (p-1)^2\,(d-1)^2/(d+2)^2$, the coefficient μ is quadratic in terms of β, and we can choose the extremum of μ with respect to β, *i.e.*,

$$\beta = \frac{(d+2)\,(d+3-p)\,\theta}{(d-1)^2\,(p-1)^2 - (d+2)^2\,(p-2)\,\theta}\;.$$

With this specific value of β, we adjust the value of θ so that $\mu(\beta) = 0$. All computations done, we find that

(9) $$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1}\quad\text{and}\quad\beta = \frac{d+2}{d+3-p}$$

and one can easily check that with these values, and without condition on θ, we actually have $\mu = 0$. When p varies in the interval $(0,2^*)$, $\theta = \theta(p)$ ranges from $\theta(1) = 0$ to $\lim_{p\to 2^*}\theta(p) = 1$ if $d\geq 2$. It is left to the reader to check that the expression of Q_g given in Section 1 coincides with Q_g^θ for the above choices of θ and β. Notice that with such a choice of θ, we get $\beta > 1$ for any $p\in(1,2^*)$ if $d\geq 3$, and any $p\in(1,5)$ if $d = 2$. On the contrary, we find that $\beta < 0$ if $p > 5$ and $d = 2$. If $p = 5$ and $d = 2$, then we have $\mu = \left(\frac{1}{\theta}-3\right)\beta^2+1$, so that θ has to be chosen strictly less than $1/3$.

Collecting the results of Section 3, with $\Lambda_\star$ given by (2), we have found that

Proposition 11. *Let $d\geq 2$, $p\in(1,2)\cup(2,2^*)$ and assume that β and θ are given by (9) if $p\neq 5$ or $d\neq 2$. If $p = 5$ and $d = 2$, we assume that $\theta\in(1/3,1)$ and $\beta = \sqrt{\theta/(3\,\theta-1)}$. With κ given by (5), we get*

$$\frac{1}{2\,\beta^2}\frac{d}{dt}\mathcal{F}[u]\leq(\lambda-\Lambda_\star)\int_{\mathfrak{M}}|\nabla u|^2\,dv_g$$

if u is a smooth positive solution of (4).

The case $d = 1$ has to be handled separately because Lemma 5 does not make sense. However, computations are much easier because

$$\int_{\mathfrak{M}}u''\frac{|u'|^2}{u}\,dv_g = \frac{1}{3}\int_{\mathfrak{M}}\frac{|u'|^4}{u^2}\,dv_g$$

so that there is no cross term to compute. The reader is invited to check that Lemmata 7, 8 and 9 hold when $d = 1$, thus showing that

$$\frac{1}{2\,\beta^2}\frac{d}{dt}\mathcal{F}[u] = -\int_{\mathfrak{M}}\left[|u''|^2 + \mu\frac{|u'|^4}{u^2}\right]\,dv_g + \lambda\int_{\mathfrak{M}}|u'|^2\,dv_g\;.$$

with $\mu := \frac{1}{3}(\kappa + \beta - 1) + \kappa(\beta - 1) = \frac{1}{3}\beta(p-1) + (1 + \beta(p-2))(\beta - 1)$. Hence, for all $p > 1$, there are two values of β, given by

$$(10) \qquad \beta_{\pm} := \frac{p - 4 \pm \sqrt{(p-1)(p+2)}}{3(p-2)},$$

such that $\mu = 0$. For $\beta = \beta_{\pm}$, we have found that

$$\frac{1}{2\beta^2} \frac{d}{dt} \mathcal{F}[u] = -\int_{\mathfrak{M}} |u''|^2 \, dv_g + \lambda \int_{\mathfrak{M}} |u'|^2 \, dv_g \leq 0$$

for any $\lambda \leq \lambda_1$.

Proposition 12. *Let $d = 1$, $p \in (1,2) \cup (2, +\infty)$. Assume that κ and β are given respectively by (5) and by (10). Then*

$$\frac{1}{2\beta^2} \frac{d}{dt} \mathcal{F}[u] \leq (\lambda - \lambda_1) \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g$$

if u is a smooth positive solution of (4) and

(i) *either $p > 2$ and $\beta \in (-\infty, \beta_-] \cup [\beta_+, +\infty)$,*
(ii) *or $p \in (1,2)$ and $\beta \in [\beta_-, 0) \cup (0, \beta_+]$.*

In order to justify the above computations, we have to establish that the solution of (4) is smooth and converges to a constant. Since this is not at all the scope of this paper, let us simply sketch that strategy of proof. A positive solution of (1) of class C^2 is actually smooth by standard elliptic theory, and thus we know that the initial datum for (4) is smooth. If we simply consider an optimal function for (3), we may proceed by regularization and approximate the solution by a bounded solution, which is also positive, bounded away from zero. A further regularization of the initial datum allows to get back to the setting of Theorems 1-3. By applying the Maximum Principle, we can then extend the boundedness properties (from above and from below) of the initial datum to the solution of (4) for any positive time. Regularity results of Harnack and Aronson-Bénilan type like the ones in [51] and subsequent papers then hold. Higher order regularity results follow: see for instance [58] in the Euclidean case. The delayed regularity phenomenon can be avoided by starting with smooth enough initial data.

It remains to show that the solution converges to a constant. One can argue as follows. Using Lemma 8 and Hölder's inequality, we know that $\int_{\mathfrak{M}} u^{2\beta} \, dv_g$ is bounded uniformly in t from above if $p > 2$ and from below if $p \in (1,2)$. By (8), is bounded uniformly in t and, as a consequence, $\int_0^{+\infty} \left(\int_{\mathfrak{M}} |\nabla u|^2 \, dv_g \right) dt$ is finite. Standard arguments allow then to prove that $\lim_{t \to +\infty} \int_{\mathfrak{M}} |\nabla u|^2 \, dv_g = 0$.

4. PROOFS

Let us consider $\Lambda_{\star}$ and θ as defined by (2) and (9).

Proposition 13. *Using the notations of Section 1, we have the estimates*

$$(1 - \theta)\lambda_1 + \theta \frac{d\rho}{d-1} \leq \lambda_{\star} \leq \Lambda_{\star} \leq \lambda_1 .$$

Proof. The first inequality is a consequence of Lemma 7. For the second one, we observe that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \|Q_g^{\theta}(1 + \varepsilon v)\|^2 = \|L_g v\|^2 .$$

Using $u = 1 + \varepsilon v$ as a test function, we find that

$$\Lambda_{\star} \leq \inf_{v \in H^2(\mathfrak{M}) \setminus \{0\}} \frac{\int_{\mathfrak{M}} (\Delta_g v)^2 \, dv_g}{\int_{\mathfrak{M}} |\nabla v|^2 \, dv_g} = \lambda_1$$

according to Lemma 5 and Lemma 7. $\qquad \square$

Proof of Theorem 3. If $v = u^{\beta}$ is a solution to (1), it is straightforward to check that

$$\Delta_g u + (\beta - 1)\frac{|\nabla u|^2}{u} - \frac{\lambda}{\beta(p-2)}(u - u^{\kappa}) = 0 .$$

Using the fact that $\int_{\mathfrak{M}} u^\kappa \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) dv_g = 0$, it follows by Proposition 11 that at $t = 0$ we have

$$0 = - \int_{\mathfrak{M}} \left(\Delta_g u + (\beta - 1) \frac{|\nabla u|^2}{u} - \frac{\lambda}{\beta (p-2)} (u - u^\kappa) \right) \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right) dv_g$$

$$= \frac{1}{2 \beta^2} \frac{d}{dt} \mathcal{F}[u] \le (\lambda - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2 dv_g \,,$$

so that u is constant if $\lambda < \Lambda_\star$. $\square$

Proof of Theorem 4. We apply Proposition 11 to $u(t = 0, \cdot) = v^{1/\beta}$. For $\lambda \le \Lambda_\star$, $\mathcal{F}[u]$ is nonincreasing. We know that as $t \to \infty$ u and ∇u converge respectively to a constant and to 0, which proves that $\mathcal{F}[u(t = 0, \cdot)] \ge \mathcal{F}[u(t, \cdot)] \ge 0 = \lim_{t \to \infty} \mathcal{F}[u(t, \cdot)]$ for any $t \ge 0$. Hence, for any $\lambda \le \Lambda_\star$, this proves that $\mathcal{F}[u]$ is nonnegative for any u, *i.e.*, that (3) holds.

If $\lambda = \Lambda_\star$ and if $v = u^\beta$ is a solution to (1), we get that

$$0 \le \mathcal{F}[u] = \|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2 - \frac{\Lambda_\star}{p-2} \left[\|v\|_{\mathrm{L}^p(\mathfrak{M})}^2 - \|v\|_{\mathrm{L}^2(\mathfrak{M})}^2 \right] = \frac{\Lambda_\star}{p-2} \left[\|v\|_{\mathrm{L}^p(\mathfrak{M})}^p - \|v\|_{\mathrm{L}^p(\mathfrak{M})}^2 \right] \,,$$

thus proving that $\|v\|_{\mathrm{L}^p(\mathfrak{M})} \ge 1$. As a special case, we get that if v is an optimal function for (3), then we can assume that $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ with no restriction and if u is the solution of (4) with initial datum $v^{1/\beta}$, then

$$\mathcal{F}[u] = \int_0^\infty \left[(1 - \theta) \int_{\mathfrak{M}} (\Delta_g u)^2 dv_g + \frac{\theta \, d}{d-1} \int_{\mathfrak{M}} \left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] dv_g - \Lambda_\star \int_{\mathfrak{M}} |\nabla u|^2 dv_g \right] = 0 \,.$$

Arguing as in Proposition 13, since $\|\mathrm{Q}_g u\|^2 \sim \|\mathrm{L}_g u\|^2$ as $t \to +\infty$ and using Lemma 5, we find that

$$0 = (1 - \theta) \int_{\mathfrak{M}} (\Delta_g u)^2 dv_g + \frac{\theta \, d}{d-1} \int_{\mathfrak{M}} \left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] dv_g - \Lambda_\star \int_{\mathfrak{M}} |\nabla u|^2 dv_g$$

$$\sim \left[(1 - \theta) \int_{\mathfrak{M}} (\Delta_g u)^2 dv_g + \frac{\theta \, d}{d-1} \int_{\mathfrak{M}} \left[\|\mathrm{L}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] dv_g - \Lambda_\star \int_{\mathfrak{M}} |\nabla u|^2 dv_g \right]$$

$$= \int_{\mathfrak{M}} (\Delta_g u)^2 dv_g - \Lambda_\star \int_{\mathfrak{M}} |\nabla u|^2 dv_g \ge (\lambda_1 - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2 dv_g \,,$$

thus obtaining an obvious contradiction if $\Lambda_\star < \lambda_1$, unless u is constant. Hence, in that case, we know that all minimizers of $\mathcal{F}$ are constants when $\lambda = \Lambda_\star$.

Now let us consider the optimal constant Λ in (3). If $\Lambda = \Lambda_\star$, consider a sequence $(\lambda_n)_{n \in \mathbb{N}}$ such that $\lambda_n > \Lambda_\star$ for any $n \in \mathbb{N}$, $\lim_{n \to \infty} \lambda_n = \Lambda_\star$ and a sequence $(v_n)_{n \in \mathbb{N}}$ of functions in $\mathrm{H}^1(\mathfrak{M})$ such that $\|v_n\|_{\mathrm{L}^2(\mathfrak{M})} = 1$ for any $n \in \mathbb{N}$ and $\mathcal{F}_{\lambda_n}[v_n] < 0$ where, with obvious notations, $\mathcal{F}_{\lambda_n}$ denotes the functional $\mathcal{F}$ written for $\lambda = \lambda_n$. Actually we can consider minimizers of $\mathcal{F}_{\lambda_n}$ in $\{ u \in \mathrm{H}^1(\mathfrak{M}) : \|u\|_{\mathrm{L}^2(\mathfrak{M})} = 1 \}$. Let us consider the quotient

$$\mathcal{Q}[v] := \frac{(p-2) \|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2}{\|v\|_{\mathrm{L}^p(\mathfrak{M})}^2 - \|v\|_{\mathrm{L}^2(\mathfrak{M})}^2}$$

and observe that the optimal constant Λ is such that

$$\Lambda = \inf_{v \in \mathrm{H}^1(\mathfrak{M}) \setminus \{0\}} \mathcal{Q}[v] = \lim_{n \to \infty} \mathcal{Q}[v_n] \,.$$

Unless $p = 2\,d/(d-2)$, $d \ge 3$, up to the extraction of a subsequence, $(v_n)_{n \in \mathbb{N}}$ converges to a limit which solves the Euler-Lagrange equation and is therefore constant. Because of the constraint $\|v_n\|_{\mathrm{L}^2(\mathfrak{M})} = 1$, the limit is 1. Let us define $w_n := \varepsilon_n (v_n - 1)$ with $\varepsilon_n := \|v_n - 1\|_{\mathrm{L}^2(\mathfrak{M})} \to 0$. A standard computation then shows that

$$\mathcal{Q}[v_n] = \mathcal{Q}[1 + \varepsilon_n w_n] \sim \frac{\|\nabla w_n\|_{\mathrm{L}^2(\mathfrak{M})}^2}{\|w_n\|_{\mathrm{L}^2(\mathfrak{M})}^2}$$

from which we deduce that $\Lambda_\star \ge \lambda_1$. Hence $\Lambda_\star < \lambda_1$ means that $\Lambda > \Lambda_\star$, which concludes the proof. $\square$

5. Improved results and consequences

Theorem 1 is a consequence of Proposition 13. The expression of $\Lambda_\star$ in Theorem 3 is not as simple as the one of $\lambda_\star$ in Theorem 2 but provides an improved condition for *rigidity*. Up to now, we have dealt only with power law nonlinearities. It is not difficult to generalize our results to nonlinearities that compare with power laws, for instance $f(v) = |v|^{p-1} v + |v|^{q-1} v$ for some $q > p$. Our first extension of Theorem 3 goes as follows.

Theorem 14. *Let f be a Lipschitz increasing function such that $f(0) = 0$ and*

$$\frac{1}{p-2}\left(f'(v) - (p-1)\frac{f(v)}{v}\right) \le 0 \quad \forall\, v > 0 \ a.e.$$

Then under the same conditions as in Theorem 3, for any $\lambda \in (0, \Lambda_\star)$, the equation

$$-\Delta_g v + \frac{\lambda}{p-2}\, v = f(v)$$

has a unique positive solution, which is equal to a constant c such that $\lambda = (p-2)\, f(c)/c$.

As a consequence, we also have the inequality

$$\|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2 - \frac{\Lambda}{p-2}\left[\left(\int_{\mathfrak{M}} p\, F(v)\, dv_g\right)^{2/p} - \|v\|_{\mathrm{L}^2(\mathfrak{M})}^2\right] \ge 0 \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

for some Λ with the same properties as in Theorem 4, and $F[v] := \int_0^v f(s)\, ds$, so that $F(v) = \frac{1}{p}\,|v|^p$ in the case of power laws.

Proof of Theorem 14. The proof is the same as the one of Theorem 3, except that

$$\int_{\mathfrak{M}} F(v)\, dv_g = \int_{\mathfrak{M}} F(u^\beta)\, dv_g$$

is not preserved by the flow defined by (4). However, the functional

$$\mathcal{F}[u] := \|\nabla(u^\beta)\|_{\mathrm{L}^2(\mathfrak{M})}^2 - \frac{\Lambda}{p-2}\left[\left(\int_{\mathfrak{M}} p\, F(u^\beta)\, dv_g\right)^{2/p} - \|u^\beta\|_{\mathrm{L}^2(\mathfrak{M})}^2\right]$$

is still non increasing because, with $v = u^\beta$, we have

$$\frac{1}{p-2}\frac{d}{dt}\int_{\mathfrak{M}} F(v)\, dv_g = \frac{\beta}{p-2}\int_{\mathfrak{M}}\left[(\beta + \kappa - 1)\frac{f(v)}{v} - \beta\, f'(v)\right]|\nabla u|^2\, dv_g \ge 0\,.$$

The proof then goes as in the power law case. $\qquad\square$

Our method actually provides an integral remainder term that can be computed using the flow, when $\lambda = \Lambda_\star$. If u a smooth solution of (4) with initial datum $v^{1/\beta}$, let

$$\mathcal{R}[v] := \beta^2 \int_0^\infty \left[(1-\theta)\int_{\mathfrak{M}} (\Delta_g u)^2\, dv_g + \frac{\theta\, d}{d-1}\int_{\mathfrak{M}}\left[\|Q_g^\theta u\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right] dv_g - \Lambda_\star \int_{\mathfrak{M}} |\nabla u|^2\, dv_g\right] dt$$

$$+ \frac{\Lambda_\star}{p-2}\int_0^\infty \left[\left(\int_{\mathfrak{M}} p\, F(u^\beta)\, dv_g\right)^{\frac{2}{p}-1}\int_{\mathfrak{M}}\left[(\beta + \kappa - 1)\frac{f(u^\beta)}{u^\beta} - \beta\, f'(u^\beta)\right]|\nabla u|^2\, dv_g\right] dt\,.$$

Corollary 15. *With the same notations as above and $\Lambda_\star$ defined by (2), for any $p \in (1,2) \cup (2, 2^*]$ if $d \ge 3$ and any $p \in (1,2) \cup (2,\infty)$ otherwise, for any smooth positive function v on $\mathfrak{M}$, we have the inequality*

$$\|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2 - \frac{\Lambda_\star}{p-2}\left[\left(\int_{\mathfrak{M}} p\, F(v)\, dv_g\right)^{2/p} - \|v\|_{\mathrm{L}^2(\mathfrak{M})}^2\right] \ge 2\, \mathcal{R}[v]\,.$$

Let us conclude this section by some comments in the case $p \in (1,2)$ and $d \ge 2$. By the definition (2) of $\Lambda_\star$, we know that

$$\Lambda_\star \ge (1-\theta)\, \lambda_1 + \theta\, \rho_\star \quad \text{with} \quad \rho_\star := \frac{d}{d-1}\inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\int_{\mathfrak{M}}\left[\|Q_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right] dv_g}{\int_{\mathfrak{M}} |\nabla u|^2\, dv_g}\,.$$

We also know that $\rho_\star \geq \rho$ and hence $\Lambda_\star \geq \lambda_\star$. Let us denote by $\Lambda(p)$ the best constant in (3). From Theorem 4 and Proposition 13, we have

$$\lambda_1 \geq \Lambda(p) \geq \Lambda_\star(p) \geq (1-\theta)\,\lambda_1 + \theta\,\rho_\star$$

with

$$\theta = \frac{(d-1)^2\,\eta}{d\,(d+2)+\eta}\,, \quad \eta = p - 1\,.$$

If $p = 1$, we have $\Lambda(1) = \lambda_1$.

On the other hand, passing to the limit as $p \to 2$ is straightforward so that we get

$$\frac{1}{2}\,\Lambda(2)\int_{\mathfrak{M}} |v|^2\,\log\left(\frac{|v|^2}{\|v\|_{\mathrm{L}^2(\mathfrak{M})}^2}\right)dv_g \leq \|\nabla v\|_{\mathrm{L}^2(\mathfrak{M})}^2 \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})\,.$$

Here $\Lambda(2)$ denotes the best constant and we have $\Lambda(2) \geq \Lambda_\star(2) \geq (1-\theta)\,\lambda_1 + \theta\,\rho_\star$ with $\theta = ((d-1)/(d+1))^2$. The interpolation method based on spectral estimates of [15] and later extended in [2, Theorem 2.4](also see [36, Section 2.2]) shows that

$$\Lambda(p) \geq \lambda_1\,\frac{1-\eta}{1-\eta^\alpha} \quad \text{with} \quad \alpha = \lambda_1/\Lambda(2) \quad \text{and} \quad \eta = p - 1\,.$$

Such an estimate is better than $\Lambda(p) \geq (1-\theta)\,\lambda_1 + \theta\,\rho_\star$ at least in a neighborhood of $p = 2$ if $\Lambda(2) > (1-\theta_2)\,\lambda_1 + \theta_2\,\rho_\star$ with $\theta_2 = ((d-1)/(d+1))^2$. This proves that for estimates on the best constant in (3), there is space for improvements.

6. CONCLUDING REMARKS

All computations in this paper are based on the action of the flow (4) on the functional $\mathcal{F}$. The functional $\mathcal{F}$ is monotonously decaying with respect to time for the right choice of the exponent β (and the optimal choice is therefore prescribed by the method) while the solution of (4) converges to a constant. The method gives a meaningful way of reinterpreting the computations of [21, 40, 11, 49] and suggest the improvements of Theorems 2, 3 and 4. However, the use of the flow itself is not mandatory except in the proof of Theorem 4 to handle the case $\lambda = \Lambda_\star$ and in the extensions corresponding to Theorem 14 and Corollary 15. The flow approach however paves the road for the various improvements presented in this paper: the integral criteria defining $\lambda_\star$ and $\Lambda_\star$ or the extension to general nonlinearities as in Theorem 14. In Theorem 4, an alternative approach could be used to directly prove that

$$\int_{\mathfrak{M}} (\Delta_g u)^2\,dv_g + \frac{\theta\,d}{d-1}\int_{\mathfrak{M}} \Big[\, \|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)\,\Big]\,dv_g - \Lambda_\star \int_{\mathfrak{M}} |\nabla u|^2\,dv_g = 0$$

means that u is a constant if $\Lambda_\star < \lambda_1$. Such a result is a consequence of our method, at least when u is a solution of (4) whose initial datum is a minimizer of $\mathcal{F}$.

The choice (9) is the one that minimizes the value of θ and therefore maximizes the value of $\Lambda_\star$. However, if we relax a little bit the condition on $\theta \in (0,1]$ and take θ such that

$$\frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} < \theta < 1\,,$$

then there is a whole interval $[\beta_-, \beta_+]$ of values of β for which the method of Section 3 still works, *i.e.*, for which μ is nonpositive, for any $p \in (1,2) \cup (2,2^*)$. If $d \geq 3$ and $p = 2^*$, it follows from (9) that $\theta = 1$ and β is uniquely defined. Consistently, we observe that $\lim_{p \to 2^*}(\beta_+(p) - \beta_-(p)) = 0$, so that the interval shrinks to a point when p approaches the critical value 2^*, $d \geq 3$. Alternatively, there are values of β for which a whole interval in terms of p can be covered, as it has been noticed in [36], in the case of the sphere when $\beta = 1$: in such a case, the method applies for any $p \in (1,2) \cup (2, 2^\sharp]$, where $2^\sharp := (2\,d^2+1)/(d-1)^2$.

Our paper leaves open the issue of deciding under which conditions the interval (Λ, λ_1) is empty or not. In other words, are there non constant minimizers of $\mathcal{F}$ for some $\lambda < \lambda_1$? If $\Lambda_* < \lambda_1$, rigidity results of Theorem 3 and optimality results of Theorem 4 are apparently of different nature, although they coincide in the case of the sphere. It is not difficult to produce examples of manifolds for which $\Lambda_* < \lambda_1$ (take non constant curvatures and consider the case p in a neighborhood of 2^* when $d \geq 3$). However, in this paper we are dealing only with sufficient conditions and it is therefore not possible to conclude that criteria for rigidity are stronger than criteria for determining the optimal constant in the interpolation inequality (3).

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